Claudio Canuto · Anita Tabacco

Mathematical Analysis I

Second Edition

Springer

Claudio Canuto
Department of Mathematical Sciences
Politecnico di Torino
Torino, Italy

Anita Tabacco
Department of Mathematical Sciences
Politecnico di Torino
Torino, Italy

UNITEXT – La Matematica per il 3+2
ISSN 2038-5722 ISSN 2038-5757 (electronic)

ISBN 978-3-319-12771-2 ISBN 978-3-319-12772-9 (eBook)
DOI 10.1007/978-3-319-12772-9
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014951876

Cover Design: Simona Colombo, Giochi di Grafica, Milano, Italy
Files provided by the Authors

Springer is a part of Springer Science+Business Media (www.springer.com)

This textbook is meant to help students acquire the basics of Calculus in curricula where mathematical tools play a crucial part (so Engineering, Physics, Computer Science and the like). The fundamental concepts and methods of Differential and Integral Calculus for functions of one real variable are presented with the primary purpose of letting students assimilate their effective employment, but with critical awareness. The general philosophy inspiring our approach has been to simplify the system of notions available prior to the university reform; at the same time we wished to maintain the rigorous exposition and avoid the trap of compiling a mere formulary of ready-to-use prescriptions.

In view of the current Programme Specifications, the organization of a first course in Mathematics often requires to make appropriate choices about lecture content, the comprehension level required from the recipients, and which kind of language to use. From this point of view, the treatise is 'stratified' in three layers, each corresponding to increasingly deeper engagement by the user. The intermediate level corresponds to the contents of the eleven chapters of the text. Notions are first presented in a naïve manner, and only later defined precisely. Their features are discussed, and computational techniques related to them are exhaustively explained. Besides this, the fundamental theorems and properties are followed by proofs, which are easily recognisable by the font's colour.

At the elementary level the proofs and the various remarks should be skipped. For the reader's sake, essential formulas, and also those judged important, have been highlighted in blue, and gray, respectively. Some tables, placed both throughout and at the end of the book, collect the most useful formulas. It was not our desire to create a hierachy-of-sorts for theorems, instead to leave the instructor free to make up his or her own mind in this respect.

The deepest-reaching level relates to the contents of the five appendices and enables the strongly motivated reader to explore further into the subject. We believe that the general objectives of the Programme Specifications are in line with the fact that willing and able pupils will build a solid knowledge, in the tradition of the best academic education. The eleven chapters contain several links to the different appendices where the reader will find complements to, and insight in various topics. In this fashion every result that is stated possesses a corresponding proof.

To make the approach to the subject less harsh, and all the more gratifying, we have chosen an informal presentation in the first two chapters, where relevant definitions and properties are typically part of the text. From the third chapter onwards they are highlighted by the layout more discernibly. Some definitions and theorems are intentionally not stated in the most general form, so to privilege a brisk understanding. For this reason a wealth of examples are routinely added along the way right after statements, and the same is true for computational techniques. Several remarks enhance the presentation by underlining, in particular, special cases and exceptions. Each chapter ends with a large number of exercises that allow one to test on the spot how solid one's knowledge is. Exercises are grouped according to the chapter's major themes and presented in increasing order of difficulty. All problems are solved, and at least half of them chaperone the reader to the solution.

We have adopted the following graphical conventions for the constituent building blocks: definitions appear on a gray background, theorems' statements on blue, a vertical coloured line marks examples, and boxed exercises, like 12. , indicate that the complete solution is provided.

We wish to dedicate this volume to Professor Guido Weiss of Washington University in St. Louis, a master in the art of teaching. Generations of students worldwide have benefited from Guido's own work as a mathematician; we hope that his own clarity is at least partly reflected in this textbook.

This second English edition reflects the latest version of the Italian book, that is in use since over a decade, and has been extensively and successfully tested at the Politecnico in Turin and in other Italian Universities. We are grateful to the many colleagues and students whose advice, suggestions and observations have allowed us to reach this result. Special thanks are due to Dr. Simon Chiossi, for the careful and effective work of translation.

Finally, we wish to thank Francesca Bonadei – Executive Editor, Mathematics and Statistics, Springer Italia – for her encouragement and support in the preparation of this textbook.

Torino, August 2014 *Claudio Canuto, Anita Tabacco*

In this introductory chapter some mathematical notions are presented rapidly, which lie at the heart of the study of Mathematical Analysis. Most should already be known to the reader, perhaps in a more thorough form than in the following presentation. Other concepts may be completely new, instead. The treatise aims at fixing much of the notation and mathematical symbols frequently used in the sequel.

We shall denote **sets** mainly by upper case letters $X, Y, \ldots$, while for the members or elements of a set lower case letters $x, y, \ldots$ will be used. When an element x is in the set X one writes $x \in X$ ('x is an element of X', or 'the element x belongs to the set X'), otherwise the symbol $x \notin X$ is used.

The majority of sets we shall consider are built starting from sets of numbers. Due to their importance, the main sets of numbers deserve special symbols, namely:

$$\begin{aligned} \mathbb{N} &= \text{set of natural numbers} \\ \mathbb{Z} &= \text{set of integer numbers} \\ \mathbb{Q} &= \text{set of rational numbers} \\ \mathbb{R} &= \text{set of real numbers} \\ \mathbb{C} &= \text{set of complex numbers.} \end{aligned}$$

The definition and main properties of these sets, apart from the last one, will be briefly recalled in Sect. 1.3. Complex numbers will be dealt with separately in Sect. 8.3.

Let us fix a non-empty set X, considered as *ambient set*. A **subset** A of X is a set all of whose elements belong to X; one writes $A \subseteq X$ ('A is contained, or included, in X') if the subset A is allowed to possibly coincide with X, and $A \subset X$ ('A is properly contained in X') in case A is a *proper* subset of X, that

C. Canuto, A. Tabacco: *Mathematical Analysis I*, 2nd Ed.,
UNITEXT – La Matematica per il 3+2 84, DOI 10.1007/978-3-319-12772-9_1,

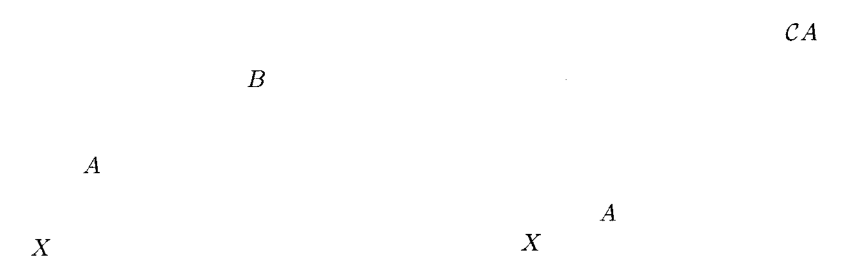

Figure 1.1. Venn diagrams (left) and complement (right)

is, if it does not exhaust the whole X. From the intuitive point of view it may be useful to represent subsets as bounded regions in the plane using the so-called *Venn diagrams* (see Fig. 1.1, left).

A subset can be described by listing the elements of X which belong to it

$$A = \{x, y, \ldots, z\};$$

the order in which elements appear is not essential. This clearly restricts the use of such notation to subsets with few elements. More often the notation

$$A = \{x \in X \mid p(x)\} \qquad \text{or} \qquad A = \{x \in X \ : \ p(x)\}$$

will be used (read 'A is the subset of elements x of X such that the condition $p(x)$ holds'); $p(x)$ denotes the *characteristic property* of the elements of the subset, i.e., the condition that is valid for the elements of the subset only, and not for other elements. For example, the subset A of natural numbers smaller or equal than 4 may be denoted

$$A = \{0, 1, 2, 3, 4\} \qquad \text{or} \qquad A = \{x \in \mathbb{N} \mid x \le 4\}.$$

The expression $p(x) =$'$x \le 4$' is an example of *predicate*, which we will return to in the following section.

The collection of all subsets of a given set X forms the **power set** of X, and is denoted by $\mathcal{P}(X)$. Obviously $X \in \mathcal{P}(X)$. Among the subsets of X there is the **empty set**, the set containing no elements. It is usually denoted by the symbol $\emptyset$, so $\emptyset \in \mathcal{P}(X)$. All other subsets of X are proper and non-empty.

Consider for instance $X = \{1, 2, 3\}$ as ambient set. Then

$$\mathcal{P}(X) = \{\, \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}.$$

Note that X contains 3 elements (it has *cardinality* 3), while $\mathcal{P}(X)$ has $8 = 2^3$ elements, hence has cardinality 8. In general if a finite set (a set with a finite number of elements) has cardinality n, the power set of X has cardinality 2^n.