

Claudio Canuto · Anita Tabacco

Mathematical Analysis I

Second Edition

 Springer

Claudio Canuto
Department of Mathematical Sciences
Politecnico di Torino
Torino, Italy

Anita Tabacco
Department of Mathematical Sciences
Politecnico di Torino
Torino, Italy

UNITEXT – La Matematica per il 3+2

ISSN 2038-5722

ISSN 2038-5757 (electronic)

ISBN 978-3-319-12771-2

ISBN 978-3-319-12772-9 (eBook)

DOI 10.1007/978-3-319-12772-9

Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014951876

© Springer International Publishing Switzerland 2015

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Cover Design: Simona Colombo, Giochi di Grafica, Milano, Italy

Files provided by the Authors

Springer is a part of Springer Science+Business Media (www.springer.com)

This textbook is meant to help students acquire the basics of Calculus in curricula where mathematical tools play a crucial part (so Engineering, Physics, Computer Science and the like). The fundamental concepts and methods of Differential and Integral Calculus for functions of one real variable are presented with the primary purpose of letting students assimilate their effective employment, but with critical awareness. The general philosophy inspiring our approach has been to simplify the system of notions available prior to the university reform; at the same time we wished to maintain the rigorous exposition and avoid the trap of compiling a mere formulary of ready-to-use prescriptions.

In view of the current Programme Specifications, the organization of a first course in Mathematics often requires to make appropriate choices about lecture content, the comprehension level required from the recipients, and which kind of language to use. From this point of view, the treatise is ‘stratified’ in three layers, each corresponding to increasingly deeper engagement by the user. The intermediate level corresponds to the contents of the eleven chapters of the text. Notions are first presented in a naïve manner, and only later defined precisely. Their features are discussed, and computational techniques related to them are exhaustively explained. Besides this, the fundamental theorems and properties are followed by proofs, which are easily recognisable by the font’s colour.

At the elementary level the proofs and the various remarks should be skipped. For the reader’s sake, essential formulas, and also those judged important, have been highlighted in blue, and gray, respectively. Some tables, placed both throughout and at the end of the book, collect the most useful formulas. It was not our desire to create a hierarchy-of-sorts for theorems, instead to leave the instructor free to make up his or her own mind in this respect.

The deepest-reaching level relates to the contents of the five appendices and enables the strongly motivated reader to explore further into the subject. We believe that the general objectives of the Programme Specifications are in line with the fact that willing and able pupils will build a solid knowledge, in the tradition of the best academic education. The eleven chapters contain several links to the different appendices where the reader will find complements to, and insight in

various topics. In this fashion every result that is stated possesses a corresponding proof.

To make the approach to the subject less harsh, and all the more gratifying, we have chosen an informal presentation in the first two chapters, where relevant definitions and properties are typically part of the text. From the third chapter onwards they are highlighted by the layout more discernibly. Some definitions and theorems are intentionally not stated in the most general form, so to privilege a brisk understanding. For this reason a wealth of examples are routinely added along the way right after statements, and the same is true for computational techniques. Several remarks enhance the presentation by underlining, in particular, special cases and exceptions. Each chapter ends with a large number of exercises that allow one to test on the spot how solid one's knowledge is. Exercises are grouped according to the chapter's major themes and presented in increasing order of difficulty. All problems are solved, and at least half of them chaperone the reader to the solution.

We have adopted the following graphical conventions for the constituent building blocks: definitions appear on a gray background, theorems' statements on blue, a vertical coloured line marks examples, and boxed exercises, like 12. , indicate that the complete solution is provided.

We wish to dedicate this volume to Professor Guido Weiss of Washington University in St. Louis, a master in the art of teaching. Generations of students worldwide have benefited from Guido's own work as a mathematician; we hope that his own clarity is at least partly reflected in this textbook.

This second English edition reflects the latest version of the Italian book, that is in use since over a decade, and has been extensively and successfully tested at the Politecnico in Turin and in other Italian Universities. We are grateful to the many colleagues and students whose advice, suggestions and observations have allowed us to reach this result. Special thanks are due to Dr. Simon Chiossi, for the careful and effective work of translation.

Finally, we wish to thank Francesca Bonadei – Executive Editor, Mathematics and Statistics, Springer Italia – for her encouragement and support in the preparation of this textbook.

1	Basic notions	1
1.1	Sets	1
1.2	Elements of mathematical logic	5
1.2.1	Connectives	5
1.2.2	Predicates	6
1.2.3	Quantifiers	7
1.3	Sets of numbers	8
1.3.1	The ordering of real numbers	12
1.3.2	Completeness of \mathbb{R}	17
1.4	Factorials and binomial coefficients	18
1.5	Cartesian product	21
1.6	Relations in the plane	23
1.7	Exercises	25
1.7.1	Solutions	26
2	Functions	31
2.1	Definitions and first examples	31
2.2	Range and pre-image	36
2.3	Surjective and injective functions; inverse function	38
2.4	Monotone functions	41
2.5	Composition of functions	43
2.5.1	Translations, rescalings, reflections	45
2.6	Elementary functions and properties	47
2.6.1	Powers	48
2.6.2	Polynomial and rational functions	50
2.6.3	Exponential and logarithmic functions	50
2.6.4	Trigonometric functions and inverses	51
2.7	Exercises	56
2.7.1	Solutions	58

3	Limits and continuity I	65
3.1	Neighbourhoods	65
3.2	Limit of a sequence	66
3.3	Limits of functions; continuity	72
3.3.1	Limits at infinity	73
3.3.2	Continuity. Limits at real points	74
3.3.3	One-sided limits; points of discontinuity	82
3.3.4	Limits of monotone functions	84
3.4	Exercises	87
3.4.1	Solutions	87
4	Limits and continuity II	89
4.1	Theorems on limits	89
4.1.1	Uniqueness and sign of the limit	89
4.1.2	Comparison theorems	91
4.1.3	Algebra of limits. Indeterminate forms of algebraic type	96
4.1.4	Substitution theorem	102
4.2	More fundamental limits. Indeterminate forms of exponential type	105
4.3	Global features of continuous maps	108
4.4	Exercises	115
4.4.1	Solutions	117
5	Local comparison of functions. Numerical sequences and series 123	
5.1	Landau symbols	123
5.2	Infinitesimal and infinite functions	130
5.3	Asymptotes	135
5.4	Further properties of sequences	137
5.5	Numerical series	141
5.5.1	Positive-term series	146
5.5.2	Alternating series	151
5.6	Exercises	154
5.6.1	Solutions	157
6	Differential calculus	169
6.1	The derivative	169
6.2	Derivatives of the elementary functions. Rules of differentiation	172
6.3	Where differentiability fails	177
6.4	Extrema and critical points	180
6.5	Theorems of Rolle, Lagrange, and Cauchy	183
6.6	First and second finite increment formulas	186
6.7	Monotone maps	188
6.8	Higher-order derivatives	190
6.9	Convexity and inflection points	192
6.9.1	Extension of the notion of convexity	195
6.10	Qualitative study of a function	196

6.10.1	Hyperbolic functions	198
6.11	The Theorem of de l'Hôpital	200
6.11.1	Applications of de l'Hôpital's theorem	202
6.12	Exercises	203
6.12.1	Solutions	207
7	Taylor expansions and applications	225
7.1	Taylor formulas	225
7.2	Expanding the elementary functions	229
7.3	Operations on Taylor expansions	236
7.4	Local behaviour of a map via its Taylor expansion	244
7.5	Exercises	248
7.5.1	Solutions	250
8	Geometry in the plane and in space	259
8.1	Polar, cylindrical, and spherical coordinates	259
8.2	Vectors in the plane and in space	262
8.2.1	Position vectors	262
8.2.2	Norm and scalar product	265
8.2.3	General vectors	270
8.3	Complex numbers	271
8.3.1	Algebraic operations	272
8.3.2	Cartesian coordinates	273
8.3.3	Trigonometric and exponential form	275
8.3.4	Powers and n th roots	277
8.3.5	Algebraic equations	279
8.4	Curves in the plane and in space	281
8.5	Functions of several variables	286
8.5.1	Continuity	286
8.5.2	Partial derivatives and gradient	288
8.6	Exercises	291
8.6.1	Solutions	294
9	Integral calculus I	301
9.1	Primitive functions and indefinite integrals	302
9.2	Rules of indefinite integration	306
9.2.1	Integrating rational maps	312
9.3	Definite integrals	319
9.4	The Cauchy integral	320
9.5	The Riemann integral	322
9.6	Properties of definite integrals	328
9.7	Integral mean value	330
9.8	The Fundamental Theorem of integral calculus	333
9.9	Rules of definite integration	338
9.9.1	Application: computation of areas	340

X Contents

9.10 Exercises	342
9.10.1 Solutions	345
10 Integral calculus II	357
10.1 Improper integrals	357
10.1.1 Unbounded domains of integration	357
10.1.2 Unbounded integrands	365
10.2 More improper integrals	369
10.3 Integrals along curves	370
10.3.1 Length of a curve and arc length	375
10.4 Integral vector calculus	378
10.5 Exercises	380
10.5.1 Solutions	382
11 Ordinary differential equations	389
11.1 General definitions	389
11.2 First order differential equations	390
11.2.1 Equations with separable variables	394
11.2.2 Linear equations	396
11.2.3 Homogeneous equations	399
11.2.4 Second order equations reducible to first order	400
11.3 Initial value problems for equations of the first order	401
11.3.1 Lipschitz functions	401
11.3.2 A criterion for solving initial value problems	404
11.4 Linear second order equations with constant coefficients	406
11.5 Exercises	412
11.5.1 Solutions	414
Appendices	425
A.1 The Principle of Mathematical Induction	427
A.2 Complements on limits and continuity	431
A.2.1 Limits	431
A.2.2 Elementary functions	435
A.2.3 Napier's number	437
A.3 Complements on the global features of continuous maps	441
A.3.1 Subsequences	441
A.3.2 Continuous functions on an interval	443
A.3.3 Uniform continuity	447

A.4 Complements on differential calculus	449
A.4.1 Derivation formulas	449
A.4.2 De l'Hôpital's Theorem	452
A.4.3 Convex functions	454
A.4.4 Taylor formulas	456
A.5 Complements on integral calculus	461
A.5.1 The Cauchy integral	461
A.5.2 The Riemann integral	462
A.5.3 Improper integrals	470
Tables and Formulas	473
Index	479

In this introductory chapter some mathematical notions are presented rapidly, which lie at the heart of the study of Mathematical Analysis. Most should already be known to the reader, perhaps in a more thorough form than in the following presentation. Other concepts may be completely new, instead. The treatise aims at fixing much of the notation and mathematical symbols frequently used in the sequel.

We shall denote **sets** mainly by upper case letters X, Y, \dots , while for the members or elements of a set lower case letters x, y, \dots will be used. When an element x is in the set X one writes $x \in X$ (' x is an element of X ', or 'the element x belongs to the set X '), otherwise the symbol $x \notin X$ is used.

The majority of sets we shall consider are built starting from sets of numbers. Due to their importance, the main sets of numbers deserve special symbols, namely:

\mathbb{N} = set of natural numbers
 \mathbb{Z} = set of integer numbers
 \mathbb{Q} = set of rational numbers
 \mathbb{R} = set of real numbers
 \mathbb{C} = set of complex numbers.

The definition and main properties of these sets, apart from the last one, will be briefly recalled in Sect. 1.3. Complex numbers will be dealt with separately in Sect. 8.3.

Let us fix a non-empty set X , considered as *ambient set*. A **subset** A of X is a set all of whose elements belong to X ; one writes $A \subseteq X$ (' A is contained, or included, in X ') if the subset A is allowed to possibly coincide with X , and $A \subset X$ (' A is properly contained in X ') in case A is a *proper* subset of X , that



Figure 1.1. Venn diagrams (left) and complement (right)

is, if it does not exhaust the whole X . From the intuitive point of view it may be useful to represent subsets as bounded regions in the plane using the so-called *Venn diagrams* (see Fig. 1.1, left).

A subset can be described by listing the elements of X which belong to it

$$A = \{x, y, \dots, z\};$$

the order in which elements appear is not essential. This clearly restricts the use of such notation to subsets with few elements. More often the notation

$$A = \{x \in X \mid p(x)\} \quad \text{or} \quad A = \{x \in X : p(x)\}$$

will be used (read ‘ A is the subset of elements x of X such that the condition $p(x)$ holds’); $p(x)$ denotes the *characteristic property* of the elements of the subset, i.e., the condition that is valid for the elements of the subset only, and not for other elements. For example, the subset A of natural numbers smaller or equal than 4 may be denoted

$$A = \{0, 1, 2, 3, 4\} \quad \text{or} \quad A = \{x \in \mathbb{N} \mid x \leq 4\}.$$

The expression $p(x) = ‘x \leq 4’$ is an example of *predicate*, which we will return to in the following section.

The collection of all subsets of a given set X forms the **power set** of X , and is denoted by $\mathcal{P}(X)$. Obviously $X \in \mathcal{P}(X)$. Among the subsets of X there is the **empty set**, the set containing no elements. It is usually denoted by the symbol \emptyset , so $\emptyset \in \mathcal{P}(X)$. All other subsets of X are proper and non-empty.

Consider for instance $X = \{1, 2, 3\}$ as ambient set. Then

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}.$$

Note that X contains 3 elements (it has *cardinality* 3), while $\mathcal{P}(X)$ has $8 = 2^3$ elements, hence has cardinality 8. In general if a finite set (a set with a finite number of elements) has cardinality n , the power set of X has cardinality 2^n .